

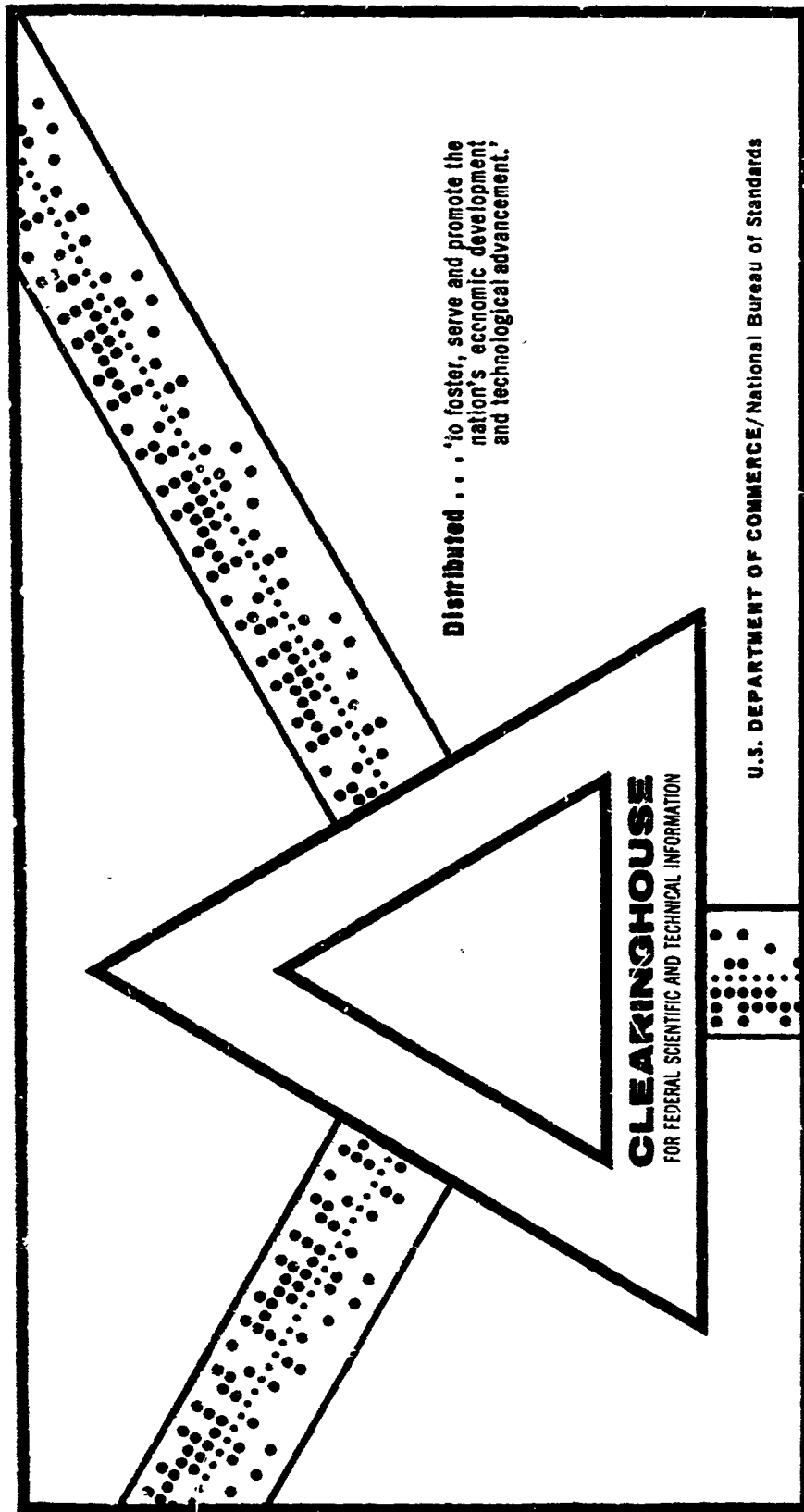
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OPTIMAL FRACTIONAL FACTORIAL PLANS FOR MAIN EFFECTS ORTHOGONAL TO TWO-FACTOR INTERACTIONS: 2 TO THE  $m$ TH POWER SERIES

J. N. Srivastava, et al

Colorado State University  
Fort Collins, Colorado

September 1969



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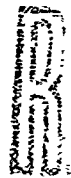
## **Aerospace Research Laboratories**

### **OPTIMAL FRACTIONAL FACTORIAL PLANS FOR MAIN EFFECTS ORTHOGONAL TO TWO-FACTOR INTERACTIONS: 2<sup>m</sup> SERIES**

J. N. SRIVASTAVA  
COLORADO STATE UNIVERSITY  
FORT COLLINS, COLORADO

D. A. ANDERSON  
UNIVERSITY OF WYOMING  
LARAMIE, WYOMING

Contract No. F33615-67-C-1436  
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FOR MAIN EFFECTS ORTHOGONAL TO  
TWO-FACTOR INTERACTIONS: 2<sup>m</sup> SERIES**

**J. N. SRIVASTAVA**  
COLORADO STATE UNIVERSITY  
FORT COLLINS, COLORADO

**D. A. ANDERSON**  
UNIVERSITY OF WYOMING  
LARAMIE, WYOMING

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UNITED STATES AIR FORCE  
WRIGHT-PATTERSON AIR FORCE BASE, OHIO**

## FOREWORD

The later part of the work done by one of the authors (J.N. Srivastava) was partly supported by the Contract F33615-67-C-1436 of the Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force. The work reported herein was partially accomplished on Project 7071, 'Research in Applied Mathematics', and was technically monitored by Dr. P. R. Krishnaiah of the Aerospace Research Labs., whose interest in the present work is greatly appreciated.

## ABSTRACT

Consider fractional factorial designs of resolution IV, i.e. where we wish to estimate only the main effects but the 2- factor interactions are not negligible. Such designs with desirable size are greatly needed both in agriculture and industry, and both in uniresponse and multiresponse experiments. The usual completely orthogonal designs involve  $N$  runs, where  $N$  is a multiple of 8. In many situations, we have a set of exactly  $N$  homogeneous experiment units, where  $N$  is not divisible by 8. For example, we may have  $N = 22$  new jet bombers of a certain kind being developed for defense purposes. Here the sampling units, namely the jets, are expensive, and furthermore cannot easily be increased in number just for the sake of our experiment. If we want to test these jets under a factorial design of resolution IV using the present available designs, then we can use only 16 of them, since 16 is the multiple of 8 nearest to but not greater than 22. This would result in a loss of  $6/22$  of the available information.

Thus the purpose of this paper is to obtain good nonorthogonal or irregular designs of resolution IV for the  $2^m$  series. Besides being of desirable size, a nonorthogonal design should be good with respect to its covariance matrix  $V$  of the estimates. In this paper, such designs (with even  $N$ ) are obtained. These designs are optimal with respect to the trace, determinant and the largest root criteria which are shown to be equivalent. In other words, among all possible designs a given value of  $N$ , our designs minimize the trace, the determinant and the largest root of  $V$ .

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In many cases, the main effects are of immediate or primary interest, but the two-factor interactions cannot be assumed negligible. "Good" designs for various symmetrical and asymmetrical factorials need therefore to be developed. The properties of "goodness" that are most important, and most often even necessary include the following: (a) The design should be of small size, i.e., the number of observations should be the desirable minimum. (b) The design should have



balance or at least partial balance. This lends facility not only to the analysis of the design, but also to the interpretation and understanding of the results.

(c) The design should further satisfy some reasonable optimality condition on the variance covariance-matrix of the estimates.

Let  $\hat{L}$  denote the (best linear unbiased) estimate of the parameter vector  $L$ . The usual variance criteria under (c) above include (i) trace, (ii) determinant, and (iii) largest root criterion. A design  $T$  is said to satisfy these criteria if, respectively, the trace, determinant, or largest root of  $V_T$  is a minimum for this design within the class of all possible designs of fixed size. Here  $V_T = \text{Cov}(\hat{L})$ , when  $L$  is estimated using  $T$ .

In this paper we shall discuss designs satisfying all the criteria mentioned above.

## 2. Some Preliminaries

We shall use a notation similar to the one in Bose and Srivastava (1964), and for the facility of the reader, repeat (without detailed proof) some basic theory of fractional designs developed therein. Treatment combinations (and also their "true effects") are denoted by  $\{a_1^{j_1} a_2^{j_2} \dots a_m^{j_m}\}$ , or equivalently  $(j_1, j_2, \dots, j_m)$  where  $j$ 's  $\in \{0,1\}$ . The  $k$  factor interactions between factors  $A_{i_1}, A_{i_2}, \dots, A_{i_k}$ , say is denoted by  $(A_{i_1} A_{i_2} \dots A_{i_k})$ . As usual, the interactions are defined, in symbolic notation by

$$(2.1) \quad A_1 A_2 \dots A_k = (a_1^1 - a_1^0)(a_2^1 - a_2^0) \dots (a_k^1 - a_k^0)(a_{k+1}^1 + a_{k+1}^0) \dots (a_m^1 + a_m^0),$$

and similarly for any set of  $k$  factors, where  $1 \leq k \leq m$ . The symbol  $\phi$ , which denotes the grand total of all  $2^m$  assemblies, is also given by (2.1), if the r.h.s. is assumed to have plus sign in each bracket.

Let  $\underline{\pi}$  denote the column vector of A's in the standard order:

$$(2.2) \quad \underline{\pi}' = [\phi; A_1, \dots, A_m; A_1 A_2, A_1 A_3, \dots, A_{m-1} A_m; A_1 A_2 A_3, \dots]$$

If  $\underline{\tau}$  ( $2^m \times 1$ ) denotes the vector of assemblies, then (2.1) implies  $\underline{\tau} = D\underline{\pi}$ , where D is a ( $2^m \times 2^m$ ) matrix, and the sum of products of the corresponding elements in any two rows of D is zero. It is easily seen that  $2^{-m/2}D$  is an orthogonal matrix and

$$(2.3) \quad \underline{\tau} = 2^{-m/2} D' \underline{\pi}.$$

Let  $\underline{\pi}$  be partitioned as

$$(2.4) \quad \underline{\pi}' = (\underline{L}' : \underline{I}_0')$$

where  $\underline{L}$  (say, ( $v \times 1$ ), where  $v = 1 + m(m+1)/2$ ) is the column vector containing all effects up to and including all the two factor interactions, and  $\underline{I}_0$  (with ( $2^m - v$ ) elements) is the vector containing all 3-factor and higher order interactions. In this paper,  $\underline{I}_0$  is assumed zero. Then (2.4) implies

$$(2.5) \quad \underline{\tau} = 2^{-m/2} D'_0 \underline{L}$$

where  $D'_0$  is the matrix obtained by cutting out the ( $2^m - v$ ) columns of  $D'$  which correspond to  $\underline{I}_0$  in (2.3).

Let T be a fractional design, i.e. a set of assemblies in which any given treatment may not occur or may occur once or more times. Let the expected values of the assemblies in T, written in the form of a vector, be denoted by  $\underline{z}^*$  where,

it is assumed there are no block effects. Let  $E'$  be the matrix obtained from  $D'_0$  by cutting out the last ( $2^m - v$ ) columns corresponding to  $\underline{I}_0$ , and also by omitting (or repeating) the rows corresponding to treatment combinations omitted (or repeated) from  $\underline{\tau}$  to get  $\underline{z}^*$ . Note that the rows of  $E'$  are arranged in such a way as to correspond to the elements of  $\underline{z}^*$ . Let  $\underline{z}$  be the vector of observations corresponding to  $\underline{z}^*$ . Assuming no block effects the model is

$$(2.6) \quad \text{Exp}(\underline{z}) = \underline{z}^* = 2^{-m} \underline{E}' \underline{L}$$

$$\text{var}(\underline{z}) = \sigma^2 \underline{I}_n$$

where  $\sigma^2$  is unknown, and  $\underline{I}_n$  is the  $(n \times n)$  identity matrix.

Then the normal equations are

$$(2.7) \quad \underline{E} \underline{E}' \hat{\underline{z}} = \underline{E} \underline{z}$$

where  $\hat{\underline{z}}$  stands for the estimate of  $\underline{z} = 2^{-m} \underline{L}$ .

Let  $T$  be a fractional design containing  $N$  assemblies. Then the symbol

$$\lambda_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_r} \quad \text{or} \quad \lambda(a_{i_1}^{j_1} a_{i_2}^{j_2} \dots a_{i_r}^{j_r})$$

denotes the number of assemblies  $\omega$  such that the symbol

$$a_{i_1}^{j_1} a_{i_2}^{j_2} \dots a_{i_r}^{j_r}$$

occurs as a part of  $\omega$ . The symbols  $a_i^j$  are to be regarded as

indeterminates and, as such, expressions like the above are operated upon, over the field of real numbers, like ordinary products of indeterminates.

Consider the set  $P$  of all polynomials with real coefficients in the symbols  $a_i^j$ ,  $(1 \leq i \leq m, j = 0, 1)$  and of degree 1 in each symbol, such that no term involves both  $a_i^0$  and  $a_i^1$ , for any  $i$ . In the sequel, we shall be concerned with only such polynomials; more information on these can be had from Srivastava (1967). Then we define

$$(2.8) \quad \lambda(\beta p_1 + \delta p_2) = \beta \lambda(p_1) + \delta \lambda(p_2),$$

for all  $p_1, p_2 \in P$ . Thus for example  $\lambda(2a_1^1 a_2^0 - 3a_1^0 a_3^1) = 2\lambda_{12}^{10} - 3\lambda_{13}^{01}$ .

If  $T_1$  and  $T_2$  are two fractional designs, then  $T_1 \oplus T_2$  shall denote the design in which any treatment combination  $\omega$  occurs  $(r_1 + r_2)$  times, provided  $\omega$  occurs respectively  $r_1$  and  $r_2$  times in  $T_1$  and  $T_2$ . Then if  $p \in P$ , we have

$$(2.9) \quad \lambda(p, T_1 \oplus T_2) = \lambda(p, T_1) + \lambda(p, T_2),$$

where  $\lambda(p, T)$  denotes the value of  $\lambda(p)$  obtained for the set of assemblies  $T$ .

Given any fraction  $T$  the matrix  $EE'$  can be directly expressed using the  $\lambda$ -operator. Every row, and hence every column, of  $EE'$  corresponds to exactly one element of  $\underline{L}$ . Indeed, the element in the  $(i, j)$  position of  $EE'$  corresponds to the elements in the  $i^{\text{th}}$  and  $j^{\text{th}}$  position of  $L$ . An element of  $\underline{L}$  can be expressed as  $A_{i_1}^{j_1} A_{i_2}^{j_2}$  where  $j_1, j_2 = 0$  or  $1$ . Then corresponding to  $\{\mu\}$  (the general mean),  $\{A_i\}$  (the main effects), and  $\{A_i A_j\}$  (the set of two-factor interactions, the matrix  $(EE')$  can be partitioned in the form

$$(2.10) \quad EE' = \begin{array}{ccc|c} \{\mu\} & \{A_i\} & \{A_i A_j\} & \\ \hline M_{11} & M_{12} & M_{13} & \{\mu\} \\ & M_{22} & M_{23} & \{A_j\} \\ \text{sym.} & & M_{33} & \{A_i A_j\} \end{array}$$

Theorem 2.1. We have, for distinct  $i, j, k$ , and  $l$ ,

$$(2.11) \quad \varepsilon(\mu) = \varepsilon(A_i, A_i) = \varepsilon(A_i A_j, A_i A_j) = N$$

$$(2.12) \quad \varepsilon(A_i, A_j) = \varepsilon(A_i, A_i A_j) = \lambda(a_i^1 - a_i^0)(a_j^1 - a_j^0)$$

$$(2.13) \quad \varepsilon(A_i, A_j) = \varepsilon(\mu, A_i A_j) = \varepsilon(A_i A_k, A_j A_k) = \lambda(a_i^1 - a_i^0)(a_j^1 - a_j^0)$$

$$(2.14) \quad \varepsilon(A_i, A_j A_k) = \lambda(a_i^1 - a_i^0)(a_j^1 - a_j^0)(a_k^1 - a_k^0)$$

$$(2.15) \quad \varepsilon(A_i A_j, A_k A_l) = \lambda(a_i^1 - a_i^0)(a_j^1 - a_j^0)(a_k^1 - a_k^0)(a_l^1 - a_l^0)$$

The value of the  $\lambda$ -operator for the above polynomials can be calculated by using the definition at (2.6). For example

$$\begin{aligned} (2.16) \quad \varepsilon(A_i, A_j A_k) &= \lambda(a_i^1 - a_i^0)(a_j^1 - a_j^0)(a_k^1 - a_k^0) \\ &= \lambda(a_i^1 a_j^1 a_k^1 - a_i^1 a_j^1 a_k^0 - a_i^1 a_j^0 a_k^1 - a_i^0 a_j^1 a_k^1 + a_i^1 a_j^0 a_k^0 + a_i^0 a_j^1 a_k^0 \\ &\quad + a_i^0 a_j^0 a_k^1 - a_i^0 a_j^0 a_k^0) \\ &= \lambda_{ijk}^{111} - \lambda_{ijk}^{110} - \lambda_{ijk}^{101} - \lambda_{ijk}^{011} + \lambda_{ijk}^{100} + \lambda_{ijk}^{010} + \lambda_{ijk}^{001} + \lambda_{ijk}^{000} \end{aligned}$$

From these equations and the definition of the  $\lambda$ -operator it can be checked that if  $T_1$  and  $T_2$  are two fractional designs, then

$$(2.17) \quad (EE')_{T_1 \oplus T_2} = (EE')_{T_1} + (EE')_{T_2},$$

for any  $T$ , where,  $(EE')_T$  is the matrix  $(EE')$  corresponding to the design  $T$ .

A fractional design  $T$  with  $N$  assemblies can be represented as an  $(m \times N)$  matrix  $T = [t_1, t_2, \dots, t_N]$ , where the column vector  $t_r$  corresponds to the  $r^{\text{th}}$  assembly  $(j_{r1}, \dots, j_{rm})'$ .

Example 2.1. As an example consider the following fractional design  $T$  for  $m=5$ .

This design is a special case of the series III designs given in section 4.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

It is easily seen using theorem 2.1 that all diagonal elements of  $M = EE'$  equal  $N (=16)$ . Further we have that  $M_{12} = 0_{15}$ , since  $\epsilon(\mu, A_i) = \lambda(a_i^1 - a_i^0) = \lambda_i^1 - \lambda_i^0 = 0$ , where  $0_{uv}$  denotes a  $(u \times v)$  matrix with zeros everywhere. Also we see that for all  $i \neq j$ ,

$$\begin{aligned} \epsilon(A_i, A_j) &= \lambda(a_i^1 - a_i^0)(a_j^1 - a_j^0) = \lambda(a_i^1 a_j^1 - a_i^1 a_j^0 - a_i^0 a_j^1 + a_i^0 a_j^0) \\ &= \lambda_{ij}^{11} - \lambda_{ij}^{10} - \lambda_{ij}^{01} + \lambda_{ij}^{00} = 0. \end{aligned}$$

Hence  $M_{22} = 16I_5$  and  $M_{13} = 0_{1,10}$ . Next we have from (2.16)

$$\epsilon(A_i, A_j, A_k) = \lambda_{ijk}^{111} - \lambda_{ijk}^{110} - \lambda_{ijk}^{101} - \lambda_{ijk}^{011} + \lambda_{ijk}^{100} + \lambda_{ijk}^{010} + \lambda_{ijk}^{001} - \lambda_{ijk}^{000}$$

which is also zero for all  $i, j$ , and  $k$ . Thus  $M_{23} = 0_{5,10}$ . Similarly, it can be

checked that for  $M_{33}$ , all the off-diagonal elements are zero, except that

$$\epsilon(A_1 A_2, A_3 A_5) = \epsilon(A_1 A_3, A_2 A_5) = \epsilon(A_1 A_5, A_2 A_3) = 16.$$

### 3. The $2^m$ Factorial Experiment

Let  $T$  be a fractional design for a  $2^m$  factorial experiment with  $N$  assemblies. Recall that  $T$  is regarded as an  $(m \times N)$  matrix of 0's and 1's where each column corresponds to an assembly.

Def. 3.1. (a)  $T$  is said to be of resolution IV, if all main effects are estimable under the assumption that 3-factor and higher order interactions are zero. (Notice that we have no such assumption regarding 2-factor interactions; they might possibly be nonzero). If, further, the estimates of the main effects have zero correlation with the estimates of every estimable linear combination of the general mean  $\mu$  and the two-factor interactions, then  $T$  is said to be of resolution IV\*.

(b) In general, in any given situation, if  $T$  is such that all parameters of interest are estimable, then  $T$  is called nonsingular. Thus, in this paper, the properties of nonsingularity and resolution IV are equivalent.

(c) The matrix  $T$  ( $m \times N$ ) is said to be  $(1,0)$  symmetric of strength  $t$  ( $\leq m$ ), if every  $(t \times N)$  submatrix remains unchanged by interchanging the symbols 0 and 1, except for the order in which the columns appear. If  $t = m$ ,  $T$  is called "completely  $(1,0)$  symmetric". (Clearly, strength  $t$  implies strength  $r$ , for  $r \leq t$ . Also, the authors have examples of arrays  $T$ , which are of a certain strength  $r$ , but not of strength  $t$ , for some  $t > r$ ).

(d)  $T$  is said to be balanced if  $\text{cov}(\hat{\mu}, \hat{A}_i)$ ,  $\text{var}(\hat{A}_i)$ , and  $\text{cov}(\hat{A}_i, \hat{A}_j)$  ( $i \neq j$ ), are independent of  $i$  and  $j$ .

Theorem 3.1.  $T$  is of resolution IV\* if and only if it is  $(1,0)$  symmetric of strength 3, and  $(EE')_T$  is nonsingular.

Proof. That  $(EE')$  be nonsingular is obviously necessary. Main effects will be orthogonal to general mean and two factor interactions if and only if  $M_{12}$  and  $M_{23}$  in equation (2.10) are zero matrices. Thus we have from (2.11)-(2.15):

$$(3.1) \quad 0 = e(\mu, A_j) = \lambda_j^1 - \lambda_j^0 = e(A_i, A_i A_j); \quad j = 1, 2, \dots, m.$$

Thus  $\lambda_i^1 = \lambda_i^0$ , for all  $i$ . Now since  $\lambda_i^\beta = \lambda_{ij}^{\beta 0} + \lambda_{ij}^{\beta 1}$  for all  $\beta$ , and all  $i \neq j$ , we

have

$$\lambda_{ij}^{11} + \lambda_{ij}^{10} = \lambda_{ij}^{00} + \lambda_{ij}^{01}, \quad \lambda_{ij}^{11} + \lambda_{ij}^{01} = \lambda_{ij}^{00} + \lambda_{ij}^{10},$$

where the second equation is obtained from the first by interchanging  $i$  and  $j$ . Hence,

$$(3.2) \quad \lambda_{ij}^{11} = \lambda_{ij}^{00} \quad \text{and} \quad \lambda_{ij}^{10} = \lambda_{ij}^{01}, \quad \text{for all } i \neq j.$$

Since  $M_{23}$  is zero,  $e(A_i, A_j, A_k)$  is zero for all distinct  $i, j$  and  $k$ . Thus, from (2.16)

$$(3.3) \quad 0 = \lambda_{ijk}^{111} + \lambda_{ijk}^{100} + \lambda_{ijk}^{010} + \lambda_{ijk}^{001} - \lambda_{ijk}^{000} - \lambda_{ijk}^{011} - \lambda_{ijk}^{101} - \lambda_{ijk}^{110};$$

As before, since  $\lambda_{ij}^{\alpha\beta} = \lambda_{ijk}^{\alpha\beta 0} + \lambda_{ijk}^{\alpha\beta 1}$ , equations (3.2) and (3.3) imply

$$(3.4) \quad \lambda_{ijk}^{111} - \lambda_{ijk}^{000} = \lambda_{ijk}^{100} - \lambda_{ijk}^{011} = \lambda_{ijk}^{010} - \lambda_{ijk}^{101} = \lambda_{ijk}^{001} - \lambda_{ijk}^{110} = 0,$$

which proves the necessity part. The sufficiency is obvious by using the above arguments in reverse.

Let  $E'$  be partitioned column-wise according to general mean, main effects, and two factor interactions,  $E' = [E'_0: E'_1: E'_2]$ . Then from (2.10),  $M_{22} = E'_1 E'_1'$ . Hence for a nonsingular  $T$ , we get from (2.7),

$$(3.5) \quad [\hat{A}_1, \hat{A}_2, \dots, \hat{A}_m]' = 2^{-m} (E'_1 E'_1')^{-1} E'_1 z$$

where  $\hat{A}_i$  is the estimate of the main effect  $A_i$ . Clearly we have

$$\text{cov}(\hat{A}_1, \hat{A}_2, \dots, \hat{A}_m)' = 2^{-2m} \sigma^2 (E'_1 E'_1')^{-1}.$$

**Theorem 3.2.** Let  $T$  satisfy the conditions of theorem 3.1. A necessary and sufficient condition that  $T$  be a balanced fractional design is that  $\lambda_{ij}^{11} = \omega$ , a constant for all  $i \neq j$ .

Proof. Suppose  $\lambda_{ij}^{11} = \omega$ , for all  $i \neq j$ . Now

$$(3.6) \quad \varepsilon(A_i, A_i) = \lambda(a_i^1 + a_i^0) = N;$$

$$(3.7) \quad \varepsilon(A_i, A_j) = \lambda(a_i^1 - a_i^0)(a_j^1 - a_j^0) = \lambda_{ij}^{11} - \lambda_{ij}^{10} - \lambda_{ij}^{01} + \lambda_{ij}^{00} = 4\omega - N.$$

Hence

$$(3.8) \quad M_{22} = (2N - 4\omega)I_m + (4\omega - N)J_{mm}; \text{ and}$$

$$(3.9) \quad M_{22}^{-1} = (g - h)I_m + hJ_{mm}, \text{ where}$$

$$(3.10a) \quad g = \frac{1}{m[4(m-1)\omega - (m-2)N]} + \frac{m-1}{m(2N-4\omega)}$$

$$(3.10b) \quad h = \frac{1}{m[4(m-1)\omega - (m-2)N]} - \frac{1}{m(2N-4\omega)}$$

$$(3.10c) \quad g - h = 1/[2N - 4\omega]$$

Hence, if  $\lambda_{ij}^{11}$  is a constant  $\omega$  for all  $i \neq j$ , then  $T$  is balanced. On the other hand if  $T$  is balanced, then  $M_{22}^{-1}$  has all diagonal elements same and all off-diagonal element same, so that  $M_{22}$  is of the form (3.8) for some value of  $\omega$ . Hence (3.6) holds for all  $i \neq j$ . This completes the proof.

It follows from the above theorem that if  $T$  is a balanced fractional design, then  $T$  is the incidence matrix of a BIBDU, (i.e. a BIBD with possibly unequal block sizes), and its parameters of the form ( $v^* = m$ ,  $b^* = N$ ,  $r^* = N/2$ , and  $\lambda^* = \omega$ ). This observation provides a simple method for obtaining balanced fractional designs. For example, if  $T_1$  is the incidence matrix of a BIBDU with parameters ( $v^* = m$ ,  $b = N/2$ ,  $r^*$ ,  $\lambda^*$ ), then  $T = T_1 \oplus \bar{T}_1$  is a balanced fractional design where  $\bar{T}_1$  is the  $(1,0)$  complement of  $T_1$ , i.e.  $\bar{T}_1$  is obtained from  $T_1$  by interchanging 1 and 0. It is obvious that  $T$  is completely  $(1,0)$  symmetric. Consider the  $i^{\text{th}}$  and  $j^{\text{th}}$  rows of  $T_1$ . Then, for  $T_1$ ,  $\lambda_{ij}^{11} = \lambda^*$  and  $\lambda_{ij}^{00} = b^* - 2r^* + \lambda^*$ , and by the method of construction of  $T$ , it is clear that  $\lambda_{ij}^{11}(T) = \lambda_{ij}^{11}(T_1) + \lambda_{ij}^{00}(T_1)$ . Hence, for  $i \neq j$  we have



$$(3.11) \quad \omega = b^* - 2(r^* - \lambda)$$

A BIBDU may be easily obtained from an ordinary BIBD, by cutting out some of the treatments and perhaps adjoining some blocks containing all treatments and/or some "blocks" containing no treatments. As an example take a (BIBD), with parameters  $(v^*, b^*, r^*, k^*, \lambda^*)$  and cut out  $v_1 < v^*$  varieties. Next adjoin  $b_1$  blocks containing each of the remaining  $(v^* - v_1)$  varieties, and  $b_2$  blocks containing no varieties. The resulting EIBDU has parameters  $((v^* - v_1), (b^* + b_1 + b_2), (r^* + b_1), (\lambda^* + b_1))$ . (It may be beneficial to the interested reader to remark here that the incidence matrix of a BIBDU is a partially balanced (PB) array of strength 2, the latter being defined for example in Chakravarti (1963) or Srivastava (1967)).

We now consider the non-singularity and optimality of the balanced fractional designs obtained in this manner.

Lemma 3.1. Let  $T$  be a balanced design as in Theorem 3.2. Then (a) The characteristic roots of  $M_{22}$  are  $[4(m-1)\omega - (m-2)N]$  and  $[2N - 4\omega]$ , with multiplicities 1 and  $(m-1)$  respectively,

(b) The characteristic roots of  $M_{22}^{-1}$  are  $[g + (m-1)h]$  and  $(g-h)$  with multiplicities 1 and  $(m-1)$  respectively. Also  $\text{tr}(M_{22}^{-1}) = mg$  and  $|M_{22}^{-1}| = [g + (m-1)h](g-h)^{m-1}$  where  $g$  and  $h$  are given by (3.10).

Proof: Let  $k_1$  and  $k_2$  be constants. Then the matrix  $k_1 I_m$  has a root  $k_1$  with multiplicity  $m$ , and the matrix  $k_2 J_{mm}$  has roots  $mk_2$  and 0 with multiplicity 1 and  $(m-1)$  respectively. Since  $I_m$  and  $J_{mm}$  commute, the roots of  $(k_1 I_m + k_2 J_{mm})$  are  $(k_1 + mk_2)$  and  $k_1$  with multiplicities 1 and  $(m-1)$ . The proof is completed by appropriate substitution for  $k_1$  and  $k_2$ , and noting that the trace and determinant are respectively the sum and the product of the roots.

Theorem 3.3. A necessary and sufficient condition that a balanced fractional design  $T$  be nonsingular is that

$$(3.12) \quad [(m-2)/4(m-1)]|T| < \omega(T) < (1/2)|T|,$$

where  $T$  denotes the number of assemblies in  $T$ , and  $\omega(T)$  is the value of  $\omega$  associated with  $T$ .

Proof: The matrix  $M_{22} = E_1 E_1'$  is obviously positive semidefinite. Thus  $M_{22}$  is nonsingular if and only if all roots of  $M_{22}$  are positive and the result follows from lemma 3.1.

Corollary 3.1. Let  $m$  be even, and let  $T$  be the incidence matrix of a BIBD with  $k^* = m/2$ , and  $v^* = m$ . Then  $T$ , considered as a fractional design is, singular. However, the balanced design obtained by adjoining to  $T$  (at least) one block containing all varieties and (at least) one block containing no variety, is nonsingular.

Proof. Since  $v^* r^* = b^* k^*$ , and  $r^*(k^*-1) = \lambda^*(v^*-1)$ , we get under the stated conditions:  $r^* = b^*/2$ ,  $|T| = b^*$ , and  $\lambda^* = \omega$ . Thus we have  $\omega = b^*(m-2)/4(m-1)$ . Hence, by (3.12),  $T$  is singular. The fractional design  $T^*$  obtained by adjoining the two blocks has  $|T^*| = b^* + 2$ , and  $\omega(T^*) = \omega + 1$ . The proof is easily completed by checking that these values satisfy (3.12).

Corollary 3.2. Let  $T = T_1 \oplus I_1$ , where  $T_1$  is the incidence matrix of a BIBDU with parameters  $v^* = m, b^*, r^*, \lambda^*$ . Then  $T$  is nonsingular if and only if

$$(3.13) \quad 0 < r^* - \lambda^* < mb^*/4(m-1).$$

Proof. Clearly  $|T| = 2b^*$ , and by (3.11),  $\omega = b^* - 2(r^* - \lambda^*)$ . Then (3.12) gives

$$2b^*(m-2)/4(m-1) < b^* - 2(r^* - \lambda^*) < 2b^*/2$$

which leads to (3.13).

Let  $\mathcal{B}$  denote the class of all balanced nonsingular completely  $(1,0)$  symmetric fractional designs. Then  $N$  is even, since  $\lambda^1 = \lambda^0$  for all  $T \in \mathcal{B}$ . Let  $\mathcal{B}_N$  denote the class of all  $T \in \mathcal{B}$  with  $|T| = N$ .

Theorem 3.4. Let  $T \in \mathcal{B}_N$ . Then  $N \geq 2m$ , and  $n_e$  (the number of degrees of freedom for error) satisfies

$$(3.14) \quad (N-2m)/2 \leq n_e \leq N-2m.$$

Proof. Let  $c = 4\omega - N$ . Then,  $M_{22} = (N-c)I_m + cJ_{mm}$ . Let  $M^*$  be the principal submatrix of  $(EE')$  having the rows and columns corresponding to  $\{\mu, A_1A_2, A_1A_3, \dots, A_1A_m\}$ . Then it can be checked by direct calculation that  $M_{22} = M^*$ . Also, since  $T \in \mathcal{B}_N$ ,  $T$  and hence  $M_{22}$  is nonsingular. Since  $M_{12}$  and  $M_{23}$  are obviously zero matrices, this implies  $\text{Rank}(EE') \geq 2m$ . But  $\text{Rank}(EE') = \text{Rank}(E) \leq N$ ; hence  $N \geq 2m$ .

Also, the number of degrees of freedom for error is  $n_e = N - \text{Rank}(M)$  and thus  $n_e \leq N - 2m$ . Finally, let  $T \in \mathcal{B}$ , and let  $(M^{**})$  be the principal submatrix of  $(M)_T$  corresponding to  $\mu$ , and 2-factor interactions. Now  $T$  can be divided into two parts  $T_1$  and  $T_2 = \bar{T}_1$ . Then it can be checked that  $M^{**}$  is the same for  $T_2$  as for  $T_1$ , except for a constant multiplier. Thus  $R(M^{**})_T = R(M^{**})_{T_1} \leq N/2$ , and

$$R(M)_T = R(M_{22})_T + R(M^{**})_T \leq m + N/2$$

Hence  $n_e \geq N - (m + N/2) = (N - 2m)/2$ , proving (3.14).

Theorem 3.5. Let  $T \in \mathcal{B}_N$ , then  $T$  is trace optimal in  $\mathcal{B}_N$  if one of the following holds for some positive integer  $\alpha$ .

$$(3.17) \quad (i) \quad N = 4\alpha \text{ and } \omega = \alpha$$

$$(3.18) \quad (ii) \quad N = 4\alpha + 2 \text{ and } \omega = \alpha + 1$$

Proof: From lemma 3.1, and (3.8) we have

$$\text{tr}(M_{22}^{-1}) = \text{tr} B = \frac{m-1}{2N-4\omega} + \frac{1}{[4(m-1)\omega - (m-2)N]} = f(\omega),$$

say, where  $f(\omega)$  is a function of  $\omega$  for fixed  $m$  and  $N$ . Suppose  $\omega$  were a continuous variable. Then

$$(3.19) \quad f'(\omega) = \frac{df(\omega)}{d\omega} = \frac{-4m(m-1)(N-4\omega)[4\omega(m-2) - (m-4)N]}{[2N-4\omega]^2[4(m-1)\omega - (m-2)N]^2}.$$

and  $f'(\omega) = 0$ , when  $\omega = N/4$  or  $\omega = (m-4)N/4(m-2)$ . The root  $\omega = (m-4)N/4(m-2)$  is extraneous since by Theorem 3.3,  $\omega > N(m-2)/4(m-1)$ . Since the denominator of (3.19) is always positive it is easy to see that  $f'(\omega) < 0$  if  $\omega < N/4$  and  $> 0$  if  $\omega > N/4$ . Hence  $f(\omega)$  attains an absolute minimum when  $\omega = N/4$  and (3.17) is proven. Now, suppose  $N = 4\alpha + 2$ . Since  $\omega$  can take only integer values,  $\omega = N/4$  is not possible. Hence we must find the integer value of  $\omega$  which makes  $f(\omega)$  as small as possible. Since  $f(\omega)$  is strictly decreasing on  $(N(m-2)/4(m-1), N/4)$  and increasing on  $(N/4, N/2)$  the absolute minimum can occur either at  $\omega = \alpha = N/4 - 1/2$  or  $\omega = \alpha + 1 = N/4 + 1/2$ . But it can be checked that

$$f(\alpha + 1) - f(\alpha) = \frac{16m(m-1)(2-m)}{(N-2)(N+2)[N+2(m-1)][N-2(m-1)]} \leq 0,$$

since  $m \geq 2$ , and  $N \geq 2m$ . This completes the proof.

Theorem 3.6. Let  $T \in \mathcal{B}_N$ , then if  $T$  is trace optimal in  $\mathcal{B}_N$ , if  $T$  is determinant optimal and conversely.

Proof. Let

$$f_1(\omega) = |M_{22}| = [4(m-1)\omega - (m-2)N][2N - 4\omega]^{m-1},$$

We must show that for any (even)  $N$ ,  $|M_{22}|$  and  $\text{tr}(M_{22}^{-1})$  become minimum for the same value of  $\omega$ .

Then

$$f_1'(\omega) = 4m(m-1)(2N-4\omega)^{m-2}(N-4\omega),$$

The roots of  $f_1'(\omega) = 0$  are  $\omega = N/2$  or  $\omega = N/4$ . Among these  $\omega = N/2$  is not in the domain of  $f_1$ . Again it is easy to see that  $f_1(\omega)$  is increasing on  $(N(m-2)/4(m-1), N/4)$  and decreasing on  $(N/4, N/2)$ ; thus an absolute maximum is reached at  $\omega = N/4$  and the theorem is proven if  $N = 4\alpha$ ,  $\alpha$  an integer. When  $N = 4\alpha + 2$ , we have

$$\begin{aligned} f_1(\alpha + 1) - f_1(\alpha) &= (N-2)^m - (N+2)^m + 2m[(N-2)^{m-1} + (N+2)^{m-1}] \\ &= (N+2)^{m-1}Q_m, \text{ say, where} \\ Q_m &= (N+2)^{m-1}[2m(1+\beta^{m-1}) - 4(1-\beta)^{-1}(1-\beta^m)], \end{aligned}$$

where  $\beta = (N-2)/(N+2)$ . Now  $Q_3 = 32(N+2) > 0$ . This proves the result for  $m=3$ , since  $f_1(\alpha+1) - f_1(\alpha) > 0$ . For larger  $n$ , it is clearly sufficient to show that  $Q_m$  is an increasing function of  $m$ . But it can be checked that

$$Q_m - Q_{m-1} = 2(1-\beta)[(1-\beta^{m-2}) + \beta(1-\beta^{m-3}) + \dots + \beta^{m-2}(1-\beta^{m-n})],$$

which is  $\geq 0$ , since  $\beta < 1$ . This completes the proof.

Theorem 3.7. Let  $T \in \mathcal{B}_N$ , then trace optimality of  $T$  implies maximum root optimality, and conversely.

Proof: From lemma 3.1 the characteristic roots of  $M_{22}^{-1}$  are  $[4(m-1)\omega - (m-2)N]^{-1}$  and  $(2N-4\omega)^{-1}$ . Now

$$(3.20) \quad [4(m-1)\omega - (m-2)N] <, =, \text{ or } > 2N-4, \text{ according as } \omega <, =, > N/4.$$

If  $N=4\alpha$ , it is obvious from (3.20) that the maximum root of  $M_{22}^{-1}$  takes the smallest value when  $\omega = N/4$ . When  $N = 4\alpha + 2$ , the max. root of  $M_{22}^{-1}$  is  $[4(m-1)\omega - (m-2)N]^{-1}$  or  $[2N - 4\omega]^{-1}$  according as  $\omega = \alpha$  or  $\alpha + 1$ .

But (3.20) shows that among these, the value of the max. root of  $M_{22}$  is smaller at  $\omega = \alpha + 1$ . This completes the proof.

Theorem 3.8. Let  $T_1$  be the incidence matrix of a BIBDU with parameters  $(v^* = m, b^*, r^*, \lambda^*)$  such that  $T = T_1 \oplus \bar{T}_1 \in \mathcal{B}_N$ . Let  $\mathcal{C}$  denote the class of fractions obtainable from BIBD's in this manner. Then  $T$  is trace optimal within  $\mathcal{C}$  if there exists an integer  $\alpha$  such that any of the following conditions hold:

- (i)  $b = 4\alpha + \theta$ ;  $\theta = 0, 1$ , or  $2$ , and  $r^* - \lambda^* = \alpha$ ;
- (ii)  $b = 4\alpha + 3$ , and  $r^* - \lambda^* = \alpha + 1$ .

Proof: It has already been shown that if  $T = T_1 \oplus \bar{T}_1$ , then  $\omega = b - 2(r^* - \lambda^*)$  and  $N = 2b^*$ . Thus  $r^* - \lambda^* = (b-\omega)/2$ .

When  $b^* = 4\alpha$ , we get  $N = 4(2\alpha)$ , and the optimum value of  $\omega$  is  $2\alpha$  so that  $r^* - \lambda^* = \alpha$ .

When  $b^* = 4\alpha + 1$ , we get  $N = 4(2\alpha) + 2$ , and the optimum value of  $\omega$  is  $(2\alpha + 1)$ ,

so that  $r^* - \lambda^* = \alpha$ . Next, suppose  $b^* = 4\alpha + 2$ . Then  $N = 4(2\alpha + 1)$  and the algebraically optimum value of  $\omega$  is  $(2\alpha + 1)$  which implies  $(r^* - \lambda^*) = (\alpha + 1/2)$ , which is not possible since  $r^*$ ,  $\lambda^*$  and  $\alpha$  are integers. Hence the combinatorially possible optimum value of  $\omega$  is either  $2\alpha$  or  $(2\alpha + 2)$ . Define  $f(\omega)$  be as in the proof of theorem 3.5. Then

$$f(2\alpha + 2) - f(2\alpha) = \frac{m}{2} \left[ \frac{-64(m-1)(m-2)}{(\frac{N}{2} - 2)[N+4(m-1)](N/2 + 2)[N-4(m-1)]} \right]$$

which is  $<$ , or  $>$  0, according as  $N >$ , or  $<$   $4(m-1)$ . But  $N < 4(m-1)$  implies  $\omega < N(m-2)/4(m-1)$ . Hence  $\omega = 2\alpha + 2$ . Thus part (i) is proven, since  $r^* - \lambda^* = [(4\alpha + 2) - (2\alpha + 2)]/2 = \alpha$ .

If  $b^* = 4\alpha + 3$ , then  $N = 4(2\alpha + 1) + 2$  and from theorem 3.5, the algebraically optimum value for  $\omega$  is  $(2\alpha + 2)$ . However,  $\omega = 2\alpha + 2$  leads to  $r^* - \lambda^* = (\alpha + 1/2)$ . Thus the choice for actual optimum is between  $\omega = (2\alpha + 1)$  or  $\omega = (2\alpha + 3)$ . The remainder of the proof is identical with the preceeding, and will be omitted to avoid repetition.

#### 4. Balanced Designs for the $2^m$ Factorial Experiment

Several balanced fractions are available for the  $2^m$  factorial experiment which permit estimates of main effects free of two factor interactions. Certain fractions will now be presented along with a note on their optimality.

Def. 4.1. If  $T_1$  and  $T_2$  are two balanced fractions with  $N$  assemblies each, the efficiency of  $T_1$  relative to  $T_2$  will be defined as

$$(4.1) \quad \text{Eff}(T_1/T_2) = [\text{tr}(EE')_{T_2}^{-1}] [\text{tr}(EE')_{T_1}^{-1}]^{-1}$$

The efficiency  $\text{Eff}(T)$ , of a balanced fraction  $T$ , is defined as the relative efficiency of  $T$  relative to a (possibly nonexistent) fraction satisfying the

optimality condition of theorem 3.4. We now present some actual designs obtainable by the method of the last section.

Series I. Let  $T = [J_{mp} : 0_{mp} : I_m : \bar{I}_m]$ , where  $\bar{I}_m$  is obtained from the matrix  $I_m$  by interchanging 0 and 1. For these fractions,  $|T| = 2(m+p)$ , and

$$\text{Eff}(T) = \frac{4[m(m+p) - 4(m-1)][2m + p - 2]^\sigma}{(m+p)^{1\sigma} (2m+p-1)^\sigma (m+p-1)^\sigma [(m+p)(m-1) - 4(m-2)]}$$

where  $\sigma$  equals 0 or 1 according as  $(m+p)$  is even or odd. It can be checked by direct calculation that the efficiency of these designs decreases as  $m$  and  $p$  increase, from about 70% for  $m = 7, p = 0$ , to about 25% for  $m = 12, p = 5$ .

As an example of a fraction constructed from balanced incomplete block designs, consider the series of BIBD with parameters  $(v^* = m = 4\lambda^* + 3, b^* = 4\lambda^* + 3, \gamma^* = 2\lambda^* + 1, K^* = 2\lambda^* + 1, \lambda^*)$ , where  $4\lambda^* + 3$  is a prime power, or a prime. Their existence is proved in Bose (1940). Let  $T_1$  be the incidence matrix of a BIBD of this form.

Series II. Let  $T = J_{m1} \oplus 0_{m1} \oplus T_1 \oplus \bar{T}_1$ . Then  $N = 4(2\lambda^* + 2)$  and  $\omega = (2\lambda^* + 2)$ . The number of d.f. for error is 1. Also, the efficiency of these fractions equals 1, so that they are optimal. It is interesting to note that if the two assemblies represented by  $J_{m1}$  and  $0_{m1}$  are omitted from the design, the efficiency is reduced to  $4(\lambda^* + 1)/(8\lambda^* + 5)$ .

Series III. Let  $2\lambda^* + 1 < m < 4\lambda^* + 3$ , and cut out  $(4\lambda^* + 3 - m)$  varieties from the BIBD given in series II. Then a design with parameters  $(m, b = 4\lambda^* + 3, r = 2\lambda^* + 1, \lambda)$  is obtained, and with  $\omega = (2\lambda^* + 1)$ . From theorem 3.5 the optimum value of  $\omega$  is  $(2\lambda^* + 2)$ , and the efficiency of the design may be computed by substitution. If a column of 1's and a column of 0's is adjoined to  $T$ , then  $N = 4(2\lambda^* + 2)$  and  $\omega = (2\lambda^* + 2)$  and the efficiency of the designs in this series becomes 1.

In the following table we give fractional designs of resolution IV for the  $2^m$  series when  $7 \leq m \leq 12$ , and for several practical values of  $N$ . To obtain the set of assemblies in the design, first write the incidence matrix  $T_1^*$  of a BIBD with parameters as given in the table, and then cut out any  $v^* - m$  rows. One such BIBD in each case is given in Cochran and Cox [(1960), pp. 469] with plan number as given in the table. Then adjoin  $(N/2 - b)$  columns of zeros to  $T_1^*$  and call this matrix  $T_1$ . The final set of assemblies is then  $T = T_1 \oplus \bar{T}_1$  where  $\bar{T}_1$  is obtained from  $T_1$  by interchanging 1's and 0's. It should be noted that there are several BIBD's, in addition to the one listed, which will give fractional designs with the same parameters. From the point of view of efficiency these designs are equivalent; however, they may differ with respect to the number of degrees of freedom for error. This aspect of the designs is too cumbersome to be studied here, except that we have  $(N - 2m)/2 \leq n_e \leq N - 2m$  by Theorem 3.4.

The efficiency given in the table for the designs is relative to the class of all balanced fractional designs. Within the class of designs of the form  $T_1 \oplus \bar{T}_1$ , where  $T_1$  is the incidence matrix of a BIBDU, the designs given are optimal.



Table 1

m = 7				m = 8			m = 9		
N	(b <sub>3</sub> r <sub>3</sub> λ)	ε	Plan	(b <sub>3</sub> r <sub>3</sub> λ)	ε	Plan	(b <sub>3</sub> r <sub>3</sub> λ)	ε	Plan
16	(7,3,1)	1.00	7						
18	(7,3,1)	1.00	7						
20	(7,3,1)	0.88	7						
22	(11,5,2)	0.94	19	(11,5,2)	0.91	19	(11,5,2)	0.85	19
24	(11,5,2)	1.00	19	(12,8,5)	1.00	13	(12,8,5)	1.00	13
26	(13,4,1)	1.00	22	(13,4,1)	1.00	22	(13,4,1)	1.00	22
28	(14,7,3)	0.93	16	(13,4,1)	0.92	22	(13,4,1)	0.92	22
30	(15,6,2)	0.98	16	(15,6,2)	0.97	16	(15,6,2)	0.96	16
32	(16,6,2)	1.00	27	(16,10,6)	1.00	29	(16,10,6)	1.00	29
34	(15,7,3)	1.00	25	(16,10,6)	1.00	29	(16,10,6)	1.00	29
36	(16,6,2)	0.95	27	(16,10,6)	0.95	29	(18,9,4)	0.94	17
38	(19,9,4)	0.99	31	(18,8,3)	0.99	11	(18,8,3)	0.98	11
40	(18,8,3)	1.00	11	(18,8,3)	1.00	11	(19,10,5)	1.00	32
m = 10				m = 11			m = 12		
N	(b <sub>3</sub> r <sub>3</sub> λ)	ε	Plan	(b <sub>3</sub> r <sub>3</sub> λ)	ε	Plan	(b <sub>3</sub> r <sub>3</sub> λ)	ε	Plan
22	(11,5,2)	0.76	19	(11,5,2)	0.57	19			
24	(11,5,2)	1.00	19	(11,5,2)	1.00	19			
26	(13,4,1)	1.00	22	(11,5,2)	1.00	19	(13,4,1)	1.00	22
28	(13,4,1)	0.91	22	(11,5,2)	0.91	22	(13,4,1)	0.91	22
30	(15,6,2)	0.94	16	(15,7,3)	0.91	25	(15,7,3)	0.88	25
32	(16,6,2)	1.00	27	(16,6,2)	1.00	27	(16,6,2)	1.00	27
34	(16,6,2)	1.00	27	(16,6,2)	1.00	27	(16,6,2)	1.00	27
36	(16,6,2)	0.94	27	(16,6,2)	0.94	27	(16,6,2)	0.94	27
38	(19,9,4)	0.97	31	(19,9,4)	0.97	31	(19,9,4)	0.94	31
40	(19,9,4)	1.00	31	(19,9,4)	1.00	31	(19,9,4)	1.00	31

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<p>Consider fractional factorial designs of resolution IV, i.e. where we wish to estimate only the main effects but the 2- factor interactions are not negligible. Such designs with desirable size are greatly needed both in agriculture and industry, and both in uniresponse and multiresponse experiments. The usual completely orthogonal designs involve N runs, where N is multiple of 8. In many situations, we have a set of exactly N homogeneous experiment units, where N is not divisible by 8. For example, we may have N = 22 new jet bombers of a certain kind being developed for defense purposes. Here the sampling units, namely the jets, are expensive, and furthermore cannot easily be increased in number just for the sake of our experiment. If we want to test these jets under a factorial design of resolution IV using the present available designs, then we can use only 16 of them, since 16 is the multiple of 8 nearest to but not</p>		

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greater than 22. This would result in a loss of 6/22 of the available information.

Thus the purpose of this paper is to obtain good nonorthogonal or irregular designs of resolution IV for the 2 series. Besides being of desirable size, a nonorthogonal design should be good with respect to its covariance matrix  $V$  of the estimates. In this paper, such designs (with even  $N$ ) are obtained. These designs are optimal with respect to the trace, determinant and the largest root criteria which are shown to be equivalent. In other words, among all possible designs a given value of  $N$ , our designs minimize the trace, the determinant and the largest root of  $V$ .

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